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## Economic Life of Equipments and Depreciation Policies

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# **ECONOMIC LIFE OF EQUIPMENTS AND DEPRECIATION POLICIES**

## **1. Introduction**

A classical topic in capital theory is the determination of the economic life of equipments. That is, given a situation in which an equipment is engaged in a productive activity, one is concerned with the determination of the length of time that the equipment should be kept in operation. This length of time, which should be set taking into account that net operating revenue decreases with time, mainly due to the fact that maintenance costs tend to increase with the age of the equipment, is what is called the economic life .

Traditionally, the subject has been treated in text-books of engineering economy and capital budgeting (cf. Bierman and Smidt, 1993; de Faro, 1979; Grant and Ireson, 1970; Oakford, 1970; and Smith, 1973). However, the traditional approach usually fails to take into account the effect of depreciation on the determination of the economic life. On the contrary, the usual approach is to select the particular depreciation policy, assuming given the economic life

A notable exception is the work of Brenner and Venezia (1983), which developed an analytical methodology that explicitly takes into account the depreciation policy on the process of determining the economic life. However, concentrating attention on changes of both the tax rate and the rate of inflation, they did not investigate the effect that a change of the depreciation policy may cause to the economic life. The purpose of the present paper is to contribute to fill this gap in the literature. Specifically, in the absence of inflation and considering both the straight-line and the constant rate policies, as well as a continuous time version of the popular sum-of-the-years-digits (SOYD) method of depreciation, it will be investigated the effect of each particular depreciation policy on the economic life.

## 2. Continuous Time Approximations to the Discrete Case

As we are interested in deriving analytical solutions, the problem will be posed in a continuous time framework. To this end, given that depreciation is effectively applied in a discrete time setting, usually in annual terms, it seems appropriate to review the most common depreciation policies. In doing so, we will have the opportunity to justify the respective continuous time approximations that will be used in the study.

### a) The Straight-Line Method

Given an economic life of  $n$  periods, an equipment whose value as new is  $P_0$ , and whose net salvage value at the end of the economic life is  $S_n \geq 0$ , the straight-line method implies that the depreciation amount at the end of the  $k$ -th period is :

$$D_k = \frac{P_0 - S_n}{n}, k = 1, 2, \dots, n \quad (1)$$

Obviously, the sum of the  $n$  parcels of depreciation satisfy the requirement of being equal to the so-called depreciable-basis,  $P_0 - S_n$ .

Under the assumption of a continuous time framework, the corresponding approximation, which was adopted in Brenner and Venezia (1983), is

$$D_t = \frac{P_0 - S_T}{T}, t \in [0, T] \quad (2)$$

where  $T$  is the economic life and, henceforth, the net salvage value  $S_t$  is assumed to be a non-increasing function of time, that is  $S'_t \leq 0$  and such that  $S_t < P_0, t > 0$ .

Notice that in the continuous time case, we must have

$$\int_0^T D_t dt = P_0 - S_T \quad (3)$$

a condition that is easily seen to be satisfied by the approximation given by (2).

### b) The Matheson Method

According to the Matheson method (cf. DeGarmo and Canada, 1973), also called the constant-rate method of depreciation, assuming that  $S_n > 0$ , the book-value of the equipment decreases at the constant periodic depreciation rate  $\hat{\mathbf{b}}$ , with  $\hat{\mathbf{b}} \in [0,1)$ , in such a way that :

$$\hat{\mathbf{P}}_k = \mathbf{P}_0 (1 - \hat{\mathbf{b}})^k, k=1,2,\dots,n \quad (4)$$

where  $\hat{\mathbf{P}}_k$  denotes the book-value at the end of the  $k$ -th period, with  $\hat{\mathbf{P}}_n$  being made equal to  $S_n$ .

As, by definition,  $\mathbf{D}_k = \hat{\mathbf{P}}_{k-1} - \hat{\mathbf{P}}_k$ , it follows then that :

$$\mathbf{D}_k = \hat{\mathbf{b}} \hat{\mathbf{P}}_{k-1} = \hat{\mathbf{b}} \mathbf{P}_0 (1 - \hat{\mathbf{b}})^{k-1}, k=1,2,\dots,n \quad (5)$$

with the constant rate  $\hat{\mathbf{b}}$  fixed in such a way that

$$\hat{\mathbf{b}} = 1 - (S_n / \mathbf{P}_0)^{1/n} \quad (6)$$

Noticing that  $\sum_{k=1}^n \hat{\mathbf{b}} \mathbf{P}_0 (1 - \hat{\mathbf{b}})^{k-1} = \mathbf{P}_0 - S_n$ , it is interesting to point out that the very popular double-declining-balance depreciation method is just a particular case of Matheson's. It suffices to set  $\hat{\mathbf{b}} = 2/n$ , which implies that, if the policy of switching to the straight-line method is not adopted (see Bierman and Smidt, 1993), we now must have the net salvage value  $S_n$  fixed in such a way that  $S_n = \mathbf{P}_0 (1 - 2/n)^n$ .

In the continuous time setting, denoting by  $\mathbf{b} > 0$  the instantaneous rate of depreciation, our approximation will require that  $d\hat{\mathbf{P}}_t / (\hat{\mathbf{P}}_t dt) = -\mathbf{b}$ . Thus :

$$\hat{P}_t = P_0 e^{-bt}, t \in [0, T] \quad (7)$$

This implies that, for a given economic life  $T$ , as we must have  $S_T = \hat{P}_T$ , the constant rate  $b$  must be fixed in such a way that :

$$b = -\frac{1}{T} \log \left( \frac{S_T}{P_0} \right) \quad (8)$$

Defining

$$D_t = b \hat{P}_t = b P_0 e^{-bt} \quad (9)$$

it is easily seen that  $\int_0^T D_t dt = P_0 - S_T$

Noticing that  $b = -\log(1 - \hat{b})$ , the corresponding continuous time approximation for the double-declining balance implies that  $b = -\log\{(T-2)/T\}$ , which requires that the economic life  $T$  be greater than 2 periods.

### c) The Sum-of-the-Years - Digits Method

An also very popular method of accelerated depreciation is the so called sum-of-the-years-digits-method (SOYD). According to this method, we have :

$$D_k = \frac{2(n-k+1)(P_0 - S_n)}{n(n+1)}, k=1,2,\dots,n \quad (10)$$

Taking into account that the condition  $\sum_{k=0}^n D_k = P_0 - S_n$  is satisfied, our continuous time approximation consists in taking

$$D_t = \frac{2(T-t)(P_0 - S_T)}{T^2}, t \in [0, T] \quad (11)$$

which also satisfies the condition  $\int_0^T D_t dt = P_0 - S_T$  and which is based in substituting the sum of the digits  $(\sum_{k=0}^n k = n(n+1)/2)$  by its continuous counterpart  $\int_0^T t dt = T^2/2$ .

At this point, it is worth noticing that Brenner and Venezia (1983) adopted the procedure of seeking to approximate both the Matheson and the sum-of-the years-digits methods of depreciation by a version of an exponential method, with a constant rate  $I$ , such that:<sup>1</sup>

$$D_t = I P_0 e^{-It} / (1 - e^{-IT}), t \in [0, T] \quad (12)$$

There are two drawbacks with the above formulation. The first is related to the fact that it implies that the asset is fully depreciated during its economic life  $T$ . As pointed out, this is not compatible with the adoption of Matheson's method, which requires  $S_T > 0$ . The second is that, in principle,  $I$  can take any positive value, regardless of the economic life  $T$ . Instead of flexibility, this renders the analysis rather artificial, as the depreciation rate, excluding the case of the double-declining method, must be a function of the economic life; which is, precisely, the variable that has to be determined in an optimal way.

### 3. The Choice of the Depreciation Policy in the Case of Fixed Economic Life

Denoting by  $d$  the instantaneous rate of interest; by  $R_t$  the net operating cash flow at time  $t$  (which is assumed to be decreasing over time, that is  $R'_t < 0$ ); and by  $j$ , with  $j \in [0,1)$ , the corporate tax rate, we have that, for a given economic life  $T$ , the present value of the cash flow associated with the operation of the equipment is equal to:

$$V_T = -P_0 + \int_0^T (1-j) R_t e^{-dt} dt + \int_0^T j D_t e^{-dt} dt + S_T e^{-dT} \quad (13)$$

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<sup>1</sup> Actually, equation (16) in their paper seems to have been missprinted, as  $I$  appears on the denominator, rather than on the numerator.

with the flow of depreciation  $D_t$  depending of the particular depreciation method that is chosen . To stress this last observation, it is convenient to denote by  $\bar{V}_T$  the part of the present value function that is independent of the depreciation policy; that is :

$$\bar{V}_T = -P_0 + S_T e^{-dT} + \int_0^T R_t e^{-dt} dt \quad (14)$$

Before proceeding with the determination of the economic life, which is achieved by the maximization of the present value function  $V_T$ , it is interesting to notice that, for a given value of  $T$ , it follows from (13) that the best depreciation policy is the one which maximizes the present value of the depreciation flow :

$$\bar{D}_T = \int_0^T D_t e^{-dt} dt \quad (15)$$

That is, as in the discrete time traditional analysis, where the economic life of the equipment is assumed to be known in advance (cf. Bierman and Smidt, 1993), the depreciation policy is chosen by selecting the method that gives the maximum value for  $\bar{D}_T$ . As, if  $d=0$ , the three considered methods are equivalent (since, then,  $\bar{D}_T = P_0 - S_T$ , in each case), we are going to focus the analysis in the more interesting and realistic situation where  $d>0$ .

If the straight-line depreciation method is adopted, the corresponding present value of the depreciation-flow is :

$$\bar{D}_{T,1} = (P_0 - S_T)(1 - e^{-dT}) / (dT) \quad (16)$$

On the other hand, for the case of the sum-of-the-years-digits method, we have:

$$\bar{D}_{T,2} = 2(P_0 - S_T)(dT + e^{-dT} - 1) / (d^2 T^2) \quad (17)$$

Thus, whether or not the net salvage value  $S_T$  is null or positive, we can conclude that :



$$\bar{D}_{T,2} - \bar{D}_{T,1} = (P_0 - S_T)(\beta + 2e^{-\beta} + \beta e^{-\beta} - 2) / (\beta^2 T^2) > 0 \quad (18)$$

if  $\beta > 0$ .<sup>2</sup>

Therefore, corroborating the traditional discrete time analysis, it follows that, for a fixed economic life of the equipment, one should always choose the SOYD method over the straight-line one.

Considering now the constant-rate of depreciation policy, which requires that the net salvage value  $S_T$  be positive and that the instantaneous rate of depreciation  $\beta$  be given by (8), we have :

$$\bar{D}_{T,3} = \beta P_0 (1 - e^{-(\beta + \beta)T}) / (\beta + \beta) \quad (19)$$

or, as a function of the net salvage value  $S_T$

$$\bar{D}_{T,3} = \log(S_T / P_0) (P_0 - S_T e^{-\beta}) / \{\log(S_T / P_0) - \beta\} \quad (19')$$

A direct comparison between  $\bar{D}_{T,1}$  and  $\bar{D}_{T,3}$ , such as the one that we have been able to do for  $\bar{D}_{T,1}$  and  $\bar{D}_{T,2}$ , appears to be rather cumbersome. However, regardless of the value of the positive constant rate  $\beta$ , as long as we set  $S_T = P_0 e^{-\beta T}$ , we can justify that  $\bar{D}_{T,3} > \bar{D}_{T,1}$  on the basis of the following arguments :

a) for  $\beta=0$ ,  $\bar{D}_{T,1} = \bar{D}_{T,3} = P_0 - S_T = P_0 (1 - e^{-\beta T})$ ;

b) as  $D'_{t,3} = -\beta^2 P_0 e^{-\beta t} < 0$ ,  $D_{t,3} = \beta P_0 e^{-\beta t} > D_{t,1} = P_0 (1 - e^{-\beta T}) / T$  for  $t < t^*$ ,

where  $t^* = -\log\{(1 - e^{-\beta T}) / \beta T\} / \beta$ , with  $D_{t,3} < D_{t,1}$  for  $t \in (t^*, T]$

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<sup>2</sup> To see this, letting  $x = \beta T$  and  $y = x + 2e^{-x} + x e^{-x} - 2$ , notice that  $y = 0$  if  $x=0$ , with  $y' = 1 - (1+x)/e^x > 0$  if  $x > 0$ .

c) for  $d > 0$ , the present value of a given nominal amount is bigger the closer it is located from the origin.

Therefore, confirming the traditional result of the discrete time analysis, (cf. Grant and Ireson, 1970 and de Faro, 1969), the two considered accelerated depreciation policies, the constant-rate and the SOYD, are always preferable to the straight-line policy, if the economic life of the equipment is fixed.

On the other hand, confirming again the discrete time analysis, when it comes to the comparison between the Matheson and the SOYD policies, there is not a dominating method. This can be seen considering the following two situations. In the first, fixing  $P_0 = 10,000$ ,  $\beta = 10\%$ ,  $\delta = 8\%$  and  $T=15$ , we have  $\bar{D}_{T,2} = 4,993.58$  and  $\bar{D}_{T,3} = 5,182.9$ . Therefore, in this situation, the constant-rate policy should be the one chosen. However, if the economic life is reduced to  $T = 10$ , keeping fixed the others parameters, we will have  $\bar{D}_{T,2} = 4,650.88$  and  $\bar{D}_{T,3} = 4,637.23$ . Accordingly, the previous choice would be reversed.

#### 4. The Impact of the Depreciation Policy on the Economic Life

In this section, selecting in each case one of the three considered depreciation policies, it will be determined, in an optimal way, the corresponding economic life. The purpose is, fixing all the other elements of the problem, to determine how the depreciation policy affects the economic life.

a) the straight-line policy.

If the straight-line method of depreciation is adopted, regardless of whether or not the salvage value  $S_T$  is null, the function to be maximized is:

$$V_T = \bar{V}_T + j (P_0 - S_T) (1 - e^{-dT}) / (dT) \quad (20)$$

with

$$V'_T = \bar{V}'_T + j \{ (P_0 - S_T) (dT + e^{-dT} - 1) - T S'_T (1 - e^{-dT}) \} / (dT^2) \quad (21)$$

where

$$\bar{V}'_T = \{(1-j)R_T + S'_T - dS_T\}e^{-dT} \quad (22)$$

Therefore, assuming that the second order condition is satisfied, denoting by  $T_1^*$  the corresponding optimal economic life, it will be the solution obtained by equating to zero the expression given by (21).

b) the case of the constant-rate policy

In the most general situation, where the constant rate of depreciation is implicitly defined via the specification of the net salvage value function  $S_t$ , which is not allowed to become null, the adoption of the Matheson policy implies that the function to be maximized is :

$$V_T = \bar{V}_T + \frac{j(S_T e^{-dT} - P_0) \log(S_T / P_0)}{dT - \log(S_T / P_0)} \quad (23)$$

with

$$V'_T = \bar{V}'_T + \frac{j}{\{dT - \log(S_T / P_0)\}^2} \{dT(S'_T e^{-dT} - P_0 S'_T / S_T) + [(S'_T - dS_T)e^{-dT}(dT - \log(S_T / P_0)) - d(S_T e^{-dT} - P_0)] \log(S_T / P_0)\} \quad (24)$$

Therefore, denoting by  $T_2^*$  the corresponding optimal economic life, it will be the solution obtained equating to zero the expression given by (24), and such that renders negative the (rather cumbersome) second derivative of the function given by (23).

On the other hand, if the constant rate of depreciation  $\beta$  is fixed a priori, what implies that the net salvage value function has to be made equal to  $S_t = P_0 e^{-bt}$ , the function to be maximized may be written as :

$$V_T = \bar{V}_T + j b P_0 (1 - e^{-(b+d)T}) / (b + d) \quad (25)$$

Therefore, in this particular situation, the optimal economic life  $T_2^*$  has to be such that :

$$(1-j) R_{T_2^*} e^{-d T_2^*} + P_0 (j b - b - d) e^{-(b+d)T_2^*} = 0 \quad (26)$$

with

$$(1-j) R'_{T_2^*} - b P_0 (j b - b - d) e^{-b T_2^*} < 0 \quad (27)$$

c) the case of the sum-of-the-years-digits policy.

Now, the function to be maximized is :

$$V_T = \bar{V}_T + \frac{2j (P_0 - S_T) (d T + e^{-d T} - 1)}{d^2 T^2} \quad (28)$$

with

$$V'_T = \bar{V}'_T + 2j \{ d T (P_0 - S_T) (1 - e^{-d T}) - [T S'_T + 2 (P_0 - S_T)] (d T + e^{-d T} - 1) \} / (d^2 T^3) \quad (29)$$

Therefore, assuming that the second order condition is satisfied, denoting by  $T_3^*$  the corresponding economic life, it will be the solution obtained equating to zero the expression given by (29).

## 5. A Numerical Comparison

An analytical comparison of the economic lives  $T_1^*$ ,  $T_2^*$  and  $T_3^*$  appears, at least in general, to be meaningless. Thus, to give at least an indication of how the choice of the depreciation policy affects the economic life of the considered equipment, we are going to limit the analysis to the numerical investigation of two very simple situations.

In the first situation it will be assumed that the net operating cash flow decreases linearly with time, in such a way that  $R_t = a - \lambda t$ ; with  $\alpha$  and  $\lambda$  being positive parameters.

It is easily seen that the corresponding non-depreciation dependent contribution to the present value function is

$$\bar{V}_T = -P_0 + S_T e^{-dT} + (1-j) \left\{ \left( a - \frac{I}{d} \right) \left( \frac{1 - e^{-dT}}{d} \right) + \frac{T I e^{-dT}}{d} \right\} \quad (30)$$

with

$$\bar{V}'_T = \{(1-j)(a - TI) + S'_T - dS_T\} e^{-dT} \quad (31)$$

On Table I, fixing  $\phi=30\%$  and  $P_0=10,000$ , and considering some particular patterns for the behavior of the net salvage value function  $S_t$ , it is presented, for each one of the three considered depreciation policies, whenever applicable, the corresponding value of the economic life  $T$ , for some selected values of the parameters  $\alpha$  and  $\lambda$ , and of the (instantaneous) interest-rate  $\delta$ .

**Table I**  
**Economic Life as a Function of the Depreciation Policy**

$\alpha$	$\lambda$	$S_t$	$\delta$ (%)	Straigh Line	Matheson	SOYD
4,000	300	0	5	12.89	-	13.00
4,000	300	0	10	12.21	-	12.41
4,000	300	0	22	9.67	-	9.86
4,000	300	$10,000 e^{-0.1t}$	5	11.36	11.53	11.52
4,000	300	$10,000 e^{-0.2t}$	10	11.45	12.37	11.79
4,000	300	$10,000 e^{-0.2t}$	20	9.15	11.81	9.90
4,000	300	1,000	10	11.86	12.08	12.05
4,000	300	1,000	30	7.29	7.59	7.46
4,000	300	5,000	10	10.43	10.47	10.54
2,000	200	1,000	10	8.00	8.34	8.30
2,000	200	1,000	25	4.56	5.18	5.06
2,000	200	1,000	40	2.01	2.76	2.65

With regard to the results presented on Table I, although we do not have sufficient evidence to generalize, there are two points that should be stressed. The first is that it appears that the adoption of the straight-line policy implies the shortest economic life. The second, is that we cannot be sure a priori of which one of the two remaining policies implies the longest economic life.

Considering now the second situation, in which it will be assumed that  $R_t = \alpha e^{-\lambda t} - m$ , with  $\alpha$ ,  $\lambda$  and  $m$  being positive parameters. In this case, concentrating attention on the particular case where  $S_t = P_0 e^{-\gamma t}$ , with  $\gamma$  being also a positive parameter, we have that :

$$\bar{V}_T = -P_0 (1 - e^{-(g+d)T}) + (1-j)\alpha (1 - e^{-(d+l)T}) / (d+l) \quad (32)$$

with

$$\bar{V}'_T = -(d+g)P_0 e^{-(g+d)T} + \alpha (1-j) e^{-(d+l)T} \quad (33)$$

Thus, if  $P_0 = 10,000$ ,  $\alpha = 6,000$ ,  $\mu = 200$ ,  $\phi = 30\%$ ,  $\delta = 10\%$  and  $\gamma = \lambda = 20\%$ , it follows that the adoption of the straight-line, the Matheson and the SOYD depreciation policies will respectively make the economic life equal to 9.64, 12.77 and 10.61 periods.<sup>3</sup> That is, and this is the point that we would like to stress, the length of the economic life may be significantly affected by the choice of the depreciation policy.

## 6. Conclusion

Extending the traditional text-book approach to the subject, proceeding along the lines of the Brenner and Venezia (1983) contribution, we have developed an analytical methodology for determining the economic life of an equipment engaged in some revenue generating operation, as a function of the depreciation policy. Considering continuous

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<sup>3</sup> Notice that, if  $\mu = 0$  and the Matheson policy is adopted, we would have the rather artificial situation where  $V_T$  would increase without bound.

time approximations for the three most popular depreciation policies, the straight-line, the Matheson and the SOYD methods, it has been shown that the choice of the particular policy may significantly affect the economic life of the equipment.

In practice, given that the economic life is usually fixed as an integer number of years, our analytical procedure may be used as a first approximation. Once the analytical solution is obtained, the actual economic life would be determined considering both the corresponding rounded-up and rounded-down integers. The one with the highest corresponding present value being chosen.

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